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A MIXED VARIATIONAL PRINCIPLE AND ITS APPLICATION TO THE NONLINEAR BENDING PROBLEM OF ORTHOTROPIC TUBES—I. DEVELOPMENT OF GENERAL THEORY AND REDUCTION TO CYLINDRICAL SHELLS

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Abstract—A procedure is presented for obtaining mixed, nonlinear variational principles for elastic shells based on the intrinsic formulation of the shell equations. The applicability of the procedure is demonstrated by developing specific principles for shells of weak curvatures and for circular cylindrical shells in regular and extended forms. Other cases are also discussed. The principles are developed within the scope of small-strain, large-rotation theory for shells under the Kirchhoff–Love hypothesis and require the availability of curvature functions for the given classes of shells. No other restrictions need be placed, except for those related to the geometries of the shells under investigation. Specifically, subject to the limitation of small extensional strains, the displacements and rotations may be large and no particular mode of shell behavior is postulated. The variational functionals basically contain the strain energy of bending and the complementary energy of the membrane force resultants. These functionals are formulated in terms of curvature and stress functions and their Euler–Lagrange equations are those of normal equilibrium, Gauss compatibility and associated boundary conditions. All may be nonlinear. Using the extended principle as a starting point, approximate principles and equations are developed in Part II for the nonlinear, nonuniform bending of orthotropic circular cylindrical tubes of finite length (extended Brazier effort). The semi-membrane approximation, with membrane-type shear deformation retained, is used in the analysis, plus some added restrictions of the Rayleigh–Ritz type on the curvature and stress fields. The results can be used for problems involving tubes subjected to various beam and shell type boundary conditions. The specific example of a clamped tube subjected to pure beam bending is calculated, using solutions of the equations for weak nonlinearity and a Rayleigh method for strong nonlinearity. Application of some of the results to the nonlinear "local buckling" analysis of a finite-length tube subjected to bending compare favorably with published results. Besides the interest in the specific problem, this demonstrates the applicability of the mixed principle for obtaining direct, approximate nonlinear solutions to useful ongoing problems, as a complement to more exact, but cumbersome, finite element or series solutions.

1. INTRODUCTION

Nonlinear analysis is important to the efficient utilization of shell structures in engineering applications and to the basic understanding of natural phenomena. Due to the small value of the ratio of the shell thickness to its surface dimensions (length L, radius R), strong nonlinearities can occur even in the elastic range, and displacements which exceed the shell thickness are very common. Nonlinearities are usually tied to the rotations of the shell elements or, equivalently, to the changes in curvature $k_{\alpha\beta} = \bar{b}_{\alpha\beta} - b_{\alpha\beta}$ of its reference surface. The extensional strains $e_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta})$ are small in the majority of engineering applications (rubber-like membranes are an exception). In the above, $b_{\alpha\beta}$ and $a_{\alpha\beta}$ are, respectively, the curvature and metric tensors of the reference surface. An overbar denotes a quantity in the deformed configuration.

The complexity of the nonlinear shell problem has led to the extensive utilization of variational techniques. These are used for obtaining approximate direct solutions and for deriving approximate forms of the basic equations. They also provide the basis for numerical algorithms. The virtual work theorem is the more basic one and is also applicable to inelastic shells. The theorem of the total potential Π of the theory of elasticity (see, for example, Washizu, 1982) is widely used in *elastic* shells, where the strain energy density U is usually taken as a function of $k_{\alpha\beta}$ and $e_{\alpha\beta}$ (in shear deformation theories, transverse shearing strains are also included. These are expressed, in turn, in terms of displacements v_α and w, which are the field variables. Extended variational principles, where some of the field equations are introduced as constraints with Lagrange multipliers, have also found important use. The Hu–Washizu principle (Washizu, 1982) and the Hellinger–Reissner principle (Reissner, 1950, 1987; Pian and Tong, 1980) are important examples from the theory of elasticity. For some recent applications of these and related principles in nonlinear shell theory, see Stumpf (1979); Schmidt and Pietraszkiewicz (1981) introduced moderate rotation theories, Pietraszkiewicz and Szwabowicz (1982) gave a large rotation Lagrangian formulation, Atluri (1983) studied finite deformations, Schmidt (1984) considered large rotation, Libai and Simmonds (1988) large deformations with one space variable, and Axelrad and Emmerling (1990) vector forms, etc. In some of these formulations, some of the strain variables are replaced with corresponding resultant stress variables $n^{\alpha\beta}$ and/or $m^{\alpha\beta}$ as the case may be.

Displacements are convenient to use in nonlinear shell theory as long as they are small (but finite). In cases of truly large displacements and rotations, the expression of Π (v_α, w) becomes extremely complicated. In addition, the displacements, referred to an unrotated configuration, lose their advantageous geometrical interpretation. The reason is that in the Lagrangian displacement formulation, the undeformed metric and *unrotated directions* are used, but the configuration with respect to which equilibrium is considered is that of the *deformed state*. For recent advances in the Lagrangian displacement formulation, see Pietraszkiewicz (1992, 1993).

At the other end of the spectrum lie the pure complementary principles which utilize the stress resultants exclusively in the formulation. These pose many difficulties in the nonlinear case. Examples of cases of nonlinear shell complementary principles were given by Washizu (1980) for the Marguerre shallow shell problem and by Libai and Simmonds (1988) for one-space-variable cases.

A method of approach to the nonlinear shell problem which foregoes the use of displacements is the stress–rotation approach. Here, the field variables are the stresses and a *finite rotation* "vector" (magnitude and direction). The rotation takes on the major nonlinearities of the problem. Variational principles which are based on this approach were given, for example, by Fraeijs de Veubeke (1972), Simmonds and Danielson (1972), Stumpf (1979), Schmidt and Pietraszkiewicz (1981), Atluri (1983) and, for the one-space-variable cases, by Libai and Simmonds (1988). They may be considered as *mixed principles* since both stresses and rotations are used in the principle and, in particular, in the energy function.

The principle to be discussed and applied here belongs to the category of mixed principles, but replaces the finite rotation with a "curvature function" ψ, which, if available, solves automatically the incremental Codazzi compatibility equations of surface theory, while the incremental Gauss equation is introduced via a Lagrange multiplier f (which turns out to be the corresponding stress function). The field variables in the principle are (ψ, f) or, if preferred, $(\psi, e_{\alpha\beta}, f)$. The latter possibility has a larger number of field variables, but can be applied (in its virtual form) to inelastic shells as well as elastic ones.

In this paper, an "incremental" form of a universally valid equation is defined to be the *exact* equation obtained by calculating its "change" due to changes in its constituent variables. Both changes are not necessarily small. See the explanation which follows eqn (5) for the "incremental" Codazzi equation. The "incremental" Gauss equation, which is derived by a similar process, is contained in eqn (17). In small extensional strain theory, quadratic terms in the *extensional strains* may be omitted. Other shell theory researchers have adopted similar equations (cf. Danielson, 1970; Koiter and Simmonds, 1973; Libai and Simmonds, 1983; Pietraszkiewicz, 1989). For further discussion, see Koiter (1966), especially pp. 19–20 and eqn (5.10).

The direct use of $k_{\alpha\beta}$ and $e_{\alpha\beta}$ as field variables places this approach within the framework of intrinsic formulations of shell equations. The literature on these formulations is extensive.

For some recent contributions, see Koiter (1980), Libai (1981, 1983), Axelrad and Emmerling (1988), and Pietraszkiewicz (1989).

The use of ψ and f is similar to the one adopted in some stress–curvature approaches to shell analysis, with the normal displacement w serving as an approximate curvature function. The latter finds its uses in nonlinear shallow shell equations and in some versions of cylindrical shell analysis. It is not designed, however, for large deformations, where the use of w is no longer valid for describing $k_{\alpha\beta}$. See the example in Part II.

It will be assumed in Part II that the extensional strains are small to the extent that nonlinear terms in these strains *and their first derivatives* can be neglected (some exceptions can be incorporated into the theory). This makes the task of finding ψ easier, since the incremental Codazzi equations then become linear in $k_{\alpha\beta}$ and in $e_{\alpha\beta}$ separately; mixed terms of the $e_{\alpha\beta}\,k_{\gamma\delta}$ type do appear in the equations, but they are usually small compared with the linear terms and can be neglected in most cases. For a more detailed proof, see the discussion following eqn (7).

It is assumed in this study that the Kirchhoff–Love assumption holds, that the strain energy density function $U(k_{\alpha\beta}, e_{\alpha\beta})$—measured per unit undeformed area of the reference surface—exists, and that the extensional strains, together with their first order derivatives, are small. Subject to the last restriction, constitutive nonlinearity can be accommodated. However, a linear orthotropic elastic material is assumed in the examples. Note that the displacements and curvature changes are not restricted in magnitude. No specific mode of shell behavior is postulated in the analysis, so that both membrane and bending modes can be accommodated.

The analysis contains several related topics:

(a) A procedure is put forth whereby, given the availability of a "curvature function" for a particular shell or class of shells, a nonlinear mixed variational principle for the shell(s) can be constructed, subject only to the small-extensional-strain approximation and otherwise unrestricted deformations. Several shell systems are discussed. The variational functional is first constructed in terms of curvature function ψ, stress function f and extensional strains $e_{\alpha\beta}$, and is then reduced to two variables. The details are provided in (b).

(b) As a prototype case, the procedure is used for a detailed derivation of the mixed principles for shells of weak curvatures (length L/mid-surface radius $R \ll 1$); see Section 2.1 for a more precise definition. Derivations of the Euler–Lagrange equations as well as nonlinear boundary conditions are included.

(c) The principle is derived for cylindrical shells, which represent the class of quasi-shallow shells (Gaussian curvature $K \approx 0$). Extensional-strain terms are included in the expressions for $k_{\alpha\beta}$. This yields a uniformly valid representation and avoids some of the pitfalls of Donnell–Mushtari–Vlasov (DMV) type theories. For an exposition of DMV theory, see Brush and Almroth (1975), chapters 5 and 6.

(d) An extended mixed variational principle for circular cylindrical shells is derived (in Part II). It incorporates mixed terms of the $k^{\alpha}_{\beta}e_{\alpha\gamma}$ type, thus facilitating the treatment of special cases such as highly bent tubes.

(e) The extended principle is applied in Part II to the approximate nonlinear analysis of finite-length orthotropic circular–cylindrical tubes subjected to nonuniform beam bending moments and shear forces. The nonlinear behavior and collapse of infinitely long isotropic cylindrical shells subjected to *pure bending* is known as the "Brazier effect." Brazier (1927) used a simplified energy approach for his analysis. Later, Reissner, in a series of papers (Reissner, 1959, 1961; Reissner and Weinitschke, 1963), developed a more exact theory, and many other studies have been made to improve on Brazier's solution. The corresponding finite-length case has been less extensively studied. Axelrad (1965, 1985), Emmerling (1984), and Axelrad and Emmerling (1983) used the flexible shell assumption for formulating equations and obtaining asymptotic and perturbation solutions. Calladine (1983) used an assumed sine function in a simplified energy analysis of the simply supported flexible shell. Antonenko (1981) studied the effect of curvature on cylindrical shell analysis. Stephens *et al.* (1975) presented numerical results using the STAGS program. The bulk of the research was confined to pure (uniform) bending. For a survey, see Axelrad and Emmerling (1984).

In the present study, the general case is formulated in terms of the mixed principle. The pure bending and nonuniform bending cases are considered. Use is then made of the semi-membrane approximation, to further simplify the formulation, but retain the effects of membrane shearing deformations.

Further approximations in the Rayleigh–Ritz sense reduce the problem to a nonlinear ordinary differential system. An approximation solution of the clamped, finite-length tube subjected to pure bending is given as an example.

Finally, it should be emphasized that the introduction is not intended as a survey of either variational principles in nonlinear shell theory or tube bending problems. Its sole purpose is to place the present study in its proper framework within these two areas of shell analysis.

2. ANALYSIS

The starting point is the theorem of the total potential Π, which is actually a stationary value theorem in nonlinear elastic problems. It states, in effect, that a necessary and sufficient condition for an equilibrium configuration to exist is $\delta\Pi = 0$, where the admissible set consists of all displacement fields which satisfy the continuity requirements everywhere in the body and on all of its boundaries.

An alternative method is to use $k_{\alpha\beta}$ and $e_{\alpha\beta}$ (tensorial or physical components) directly as field variables. This "intrinsic" formulation avoids the problematics of unwieldy strain–displacement relations in the case of large deformations, and facilitates the direct use of the small-strain approximation. However, in order to change to the intrinsic form, two requirements must be met. (a) The potential of the external loads P must be expressible in terms of $e_{\alpha\beta}$, $k_{\alpha\beta}$. This point will be discussed later (see also Koiter, 1980). (b) To assure the continuity of the displacements, $k_{\alpha\beta}$, $e_{\alpha\beta}$ must satisfy the compatibility equations of surface geometry. These are conveniently expressed in terms of the Codazzi and Gauss equations of the surface, written in an incremental form (other forms of the continuity equations are also available). Let $L_\alpha(e_{\gamma\delta}, k_{\gamma\delta})$, $L_3(e_{\gamma\delta}, k_{\gamma\delta})$ denote the *nonlinear* Codazzi and Gauss incremental compatibility equations, respectively. These can be appended to Π as constraints, with Lagrange multipliers f^α and f, yielding the *enhanced potential*:

$$\Pi_1(e_{\alpha\beta}, k_{\alpha\beta}, f^\alpha, f) = \iint_A (U - f^\alpha L_\alpha - f L_3)\,\mathrm{d}A + P \tag{1}$$

(f^α and f can be shown to constitute "stress functions" for the problem at hand). The number of independent fields is now nine. While such an enhanced form may be useful for some numerical methods, it should be useful to reduce it. One avenue of approach is to equate to zero the coefficients of $\delta e_{\alpha\beta}$ and $\delta k_{\alpha\beta}$ in the expression for $\delta\Pi_1$. This provides six equations expressing $e_{\alpha\beta}$ and $k_{\alpha\beta}$ in terms of (f^α, f). If $e_{\alpha\beta}$ and $k_{\alpha\beta}$ can be extracted in the above and resubstituted into the function, a stress function formulation $\Pi_1 = \Pi_1\ (f^\alpha, f)$ would result. While this reduction is not feasible in the general case of large rotations *and large strains*, its partial use in small-strain, finite-rotation theories has not been sufficiently explored. Also, the use of Π as an extended intrinsic principle for numerical use and for special purposes may be explored. A second method is to find general solutions to some, but not all, of the compatibility equations, and thereby reduce the number of independent fields. The latter approach is adopted in Part II.

A scalar function ψ such that $k_{\alpha\beta}\ (\psi,\ e_{\gamma\delta})$ satisfies the Codazzi incremental equations identically (exactly or approximately) is termed a "curvature function." Substitution of $k_{\alpha\beta}$ into Π_1 yields

$$\Pi_2(e_{\alpha\beta}, \psi, f) = \iint_A (U - f L_3)\,\mathrm{d}A + P, \tag{2}$$

with five independent fields. The virtual form of the above can be used for inelastic shells.

For elastic shells, a further reduction can be achieved by equating to zero the coefficients of $\delta e_{\alpha\beta}$ in the variational equation $\delta\Pi_2 = 0$. This forms three equations for $e_{\alpha\beta}$ in terms of f (and ψ). Back substitution into Π_2 yields

$$\Pi_3 = \Pi_2[e_{\alpha\beta}(\psi, f), \psi, f], \tag{3}$$

which has ψ and f as *two* independent fields.

Regarding the external load potential, it will be transformed in Part II via partial integration into a mixed/complementary potential P^*. Hence, a detailed discussion of possible forms of P will not be presented here.

The process depends on the ability to find ψ for the given problem. Emphasis is put on curvature functions which satisfy the (incremental) Codazzi equation:†

$$\varepsilon^{\beta\gamma}[k_{\alpha\beta}|_\gamma - \beta^\theta_{\alpha\gamma}(k_{\theta\beta} + b_{\theta\beta})] = 0. \tag{4}$$

Here, $\varepsilon^{\beta\gamma}$ is the surface permutation tensor, the vertical bar denotes covariant differentiation with respect to the *undeformed* geometry and $\beta^\theta_{\alpha\beta}$ is the (incremental) Christoffel-symbol tensor, which, for small strains, is given by

$$\beta^\theta_{\alpha\gamma} = e^\theta_\alpha|_\gamma + e^\theta_\gamma|_\alpha - e_{\alpha\gamma}|^\theta. \tag{5}$$

Equation (4) can be obtained directly from the universally valid Codazzi equation $\varepsilon^{\beta\gamma}(b_{\alpha\beta,\gamma} - \Gamma^\theta_{\alpha\gamma} b_{\theta\beta}) = 0$ by replacing $b_{\alpha\beta}$ with $(b_{\alpha\beta} + k_{\alpha\beta})$, $\Gamma^\theta_{\alpha\beta}$ with $(\Gamma^\theta_{\alpha\beta} + \beta^\theta_{\alpha\gamma})$, and then subtracting the original equation from the result.

Herein and in Part II, tensorial operations will be related to the undeformed geometry. If the deformation is inextensional (or nearly so), then

$$\varepsilon^{\beta\gamma} k_{\alpha\beta}|_\gamma = 0. \tag{6}$$

Let the general solution of these two *linear* equations be $k^*_{\alpha\beta}(\psi)$. The solution for the case of small extensional strain is

$$k_{\alpha\beta} \cong k^*_{\alpha\beta}(\psi) + f_{\alpha\beta}(e_{\gamma\theta}, \psi) \quad (\lim f_{\alpha\beta} = 0)$$
$$e_{\gamma\delta} \to 0. \tag{7}$$

The dependence of $f_{\alpha\beta}$ on ψ is due to the mixed (nonlinear) terms $\beta^\theta_{\alpha\gamma}$ and $k_{\theta\beta}$ in the Codazzi equations.

Within the scope of the small extensional strain approximation, it is now shown that these terms are usually small compared with other terms in the *Codazzi equations*, and thus may be normally suppressed *there* (exceptional cases will be discussed later). *No other qualifications as to the type of shell theory are needed.* For purposes of this discussion, the extensional strains are assigned the order of magnitude ε and other terms assigned the order ε^α, such that $0 \leqslant \alpha \leqslant 1$. Thus, $0 < \alpha < 1$ is fractional order and $\alpha = 0$ is order unity. The small extensional strain approximation (as used in this paper) is restated to imply that terms with $\alpha > 1$ are suppressed in the field equations.

Nondimensional terms $Rk_{\alpha\beta}$ (where R is a length measure of the undeformed surface) may assume differing values of $\alpha \leqslant 1$ which can even coexist in the same shell problem. (a) In (mostly) membrane regions away from discontinuities, $\alpha \sim 1$. (b) Fractional orders (mostly $\alpha = \frac{1}{2}$) occur in many moderate rotation theories. (c) Finally, $\alpha = 0$ may occur in strong inextensional deformations, near discontinuities, in some postbuckling problems, etc. Evidently, if $\alpha = 0$ in one region of the shell, $f_{\alpha\beta}$ can be suppressed there, but an across-the-board suppression can lead to erroneous results in other regions of the shell.

In cases (a) and (b), the $\beta^\theta_{\alpha\gamma} k_{\theta\beta}$ terms are of combined order $\alpha > 1$ and can be suppressed according to the small extensional strain approximation. In case (c), they are of *order 1*,

† For another approach to curvature functions, which is more general, but also more complicated, see Libai (1967).

but in this instance, they can be suppressed compared with the $k_{\alpha\beta}$ terms, which are now of *order zero*. It follows that the $\beta_{\alpha\gamma}k_{\theta\beta}$ terms can be suppressed *for the entire range* of $k_{\alpha\beta}$, and eqn (7) can be simplified to read

$$k_{\alpha\beta} = k^*_{\alpha\beta} + f_{\alpha\beta}(e_{\gamma\theta}). \tag{7a}$$

Koiter (1966) suggested the use of this form of the Codazzi equations in nonlinear theory; see eqn (5.10) in his paper.

Exceptional situations may occur in case (c) but *these can affect only the "membrane correction" term* $f_{\alpha\beta}$. For example, in the strong nonlinear bending of long tubes (Brazier-type problems), the dominant circumferential curvature change k_{ss} is of order $\alpha = 0$, *and can be determined by eqn (7a).* However, due to the long-shell effect, the secondary longitudinal curvature k_{xx} is of order $\alpha = 1$, so that the term $k_{ss}e_{xx}$ may be retained in the expression for f_{xx} for improved accuracy in the expression for k_{xx}. This modified procedure will be used in Part II in the section on nonlinear bending of long tubes and may be regarded as a part of the basic procedure presented herein. Another case where the retention of mixed terms should be considered is in "two-scale" problems, where smaller strains are superposed on "large" initial stress resultants and strains.

Some geometrically nonlinear shell theories retain the mixed terms $(\beta^{\theta}_{\alpha\gamma}k_{\theta\beta})$ in the Codazzi equations. As examples, see Danielson (1970), Koiter and Simmonds (1973), Libai and Simmonds (1983), and Pietraszkiewicz (1989). However, as has been shown here, a full exploitation of the small extensional strain approximation makes this retention unnecessary except for special cases. It was Koiter (1966) who first deleted the mixed terms from the Codazzi equations, in addition to the quadratic extensional strain terms; see p. 20 of his paper for eqn (5.10) and the accompanying discussion. The reduction of $k_{\alpha\beta}$ to a linear combination of ψ and $e_{\alpha\beta}$ bears only a *superficial* resemblance to the linearization of the curvature–displacement relations, which is common to several geometrically nonlinear theories, as discussed in Sanders (1963), since both reduce to linear expressions. However, in those instances, the displacements must be small (but finite) and the rotations moderate, whereas in the present case both ψ and $e_{\alpha\beta}$ may be highly nonlinear in the displacements; ψ is unrestricted and $e_{\alpha\beta}$ must be of order $\alpha = 1$, but its second derivatives may be of order $\alpha = 0$. This makes the present approach highly suitable for "large-rotation" analysis, subject to the restrictions imposed by the small extensional strain approximation.

In regions of large deformations ($\alpha = 0$), approximations of the order of ε for $f_{\alpha\beta}$ are sometimes acceptable. These may range from minor approximations to total suppression as in shallow shells and the DMV theory. In the latter situations, the normal displacement w plays the role of a curvature function such as ψ, but this holds only for small displacements.

To sum up this approach: since the general solution of the *three* nonlinear compatibility equations is, in general, impossible to obtain, the approach adopted here is to obtain solutions to two of the equations, and solve the third one in a variational sense by appending it to the total potential. This third (Gauss) equation intermixes $e_{\alpha\beta}$ and ψ, so that $e_{\alpha\beta}$ depends on ψ. This does *not* represent pure inextensional deformations. The latter would have required the deletion of $e_{\alpha\beta}$ from *all three* compatibility equations.

The form of $k^*_{\alpha\beta}$ for some classes of shells is as follows:

(a) *Developable shells* (cylindrical and conical surfaces, etc.): $k^*_{\alpha\beta} = \psi|_{\alpha\beta}$. This is extended to shells of weak Gaussian curvatures (including shallow shells, DMV theory). Also included are general shells with deformation patterns of wavelength λ, such that $K\lambda^2 \ll 1$. Here, K is the Gaussian curvature of the undeformed middle surface.

(b) *Spherical shells*: $k^*_{\alpha\beta} = \psi|_{\alpha\beta} + Ka_{\alpha\beta}\psi$. This form, with extensions to the nonlinear analysis of shells of slowly varying curvature K, was introduced by Libai (1962). For further applications, see Libai (1967), Lukasiewicz (1971) and Pietraszkiewicz (1989). For the mathematically similar form of a membrane stress function, see Finzi (1934), Langhaar (1953), Truesdell (1960), Duddeck (1964), etc.

$f_{\alpha\beta}$ are any particular symmetric integrals of the equations

$$\varepsilon^{\beta\gamma}[f_{\alpha\beta}|_{\gamma} - b^{\theta}_{\beta}(e_{\theta\alpha}|_{\gamma} + e_{\theta\gamma}|_{\alpha} - e_{\alpha\gamma}|_{\theta})] = 0. \tag{8}$$

If the $b_{\alpha\beta}$ are constants, they can be put inside the derivative signs and rearrangement produces the more convenient "divergence form":

$$\varepsilon^{\beta\gamma}(f_{\alpha\beta} - b^{\mu}_{\mu}e_{\alpha\beta} + e_{\alpha\beta}B)|_{\gamma} = 0, \tag{9}$$

where

$$B = \varepsilon^{\mu\nu}b^{\lambda}_{\mu}e_{\lambda\nu} \tag{10}$$

is an invariant which vanishes for spherical shells and also if the principal directions for strain and curvature coincide. In other instances, integral forms can be used (see example for cylindrical shells). Equation (9) can be used for shells with slowly varying curvatures. Extension to more general shells can also be considered, since this involves neglecting terms $O[(1/R^2)e_{\alpha\beta}]$ compared with $O(k_{\alpha\beta,\gamma})$, which is an acceptable approximation in most shell theories.

The following are some specific cases:

(a) *Shells of weak curvatures* (plates, shallow shells, DMV theory). Here $f_{\alpha\beta} \cong 0$.

(b) *Circular cylindrical shells*. Here, the undeformed metric is Cartesian with respect to the generators (x-direction) and hoop circles (s-direction). The mean radius of the cylinder is "a". There is no distinction between tensorial and physical components and covariant derivatives reduce to partial derivatives. The equations are

$$\left(k_{xx} - \frac{1}{a}e_{xx}\right)_{,s} - \left(k_{xs} - \frac{1}{a}\gamma_{xs}\right)_{,x} = 0. \tag{11}$$

$$\left(k_{ss} - \frac{1}{a}e_{ss}\right)_{,x} - k_{xs,s} = 0. \tag{12}$$

The solution for $f_{\alpha\beta}$ is not unique. Some possible choices are:

(b1)

$$f_{xx} = 0;\ f_{xs} = \frac{1}{a}\left(\gamma_{xs} - \int e_{xx,s}\,\mathrm{d}x\right) = -\frac{1}{a}\int \lambda\,\mathrm{d}x;$$

$$f_{ss} = \frac{1}{a}e_{ss} + \int f_{xs,s}\,\mathrm{d}x. \tag{13a}$$

Here, $\lambda = e_{xx,s} - \gamma_{xs,x}$ is the geodesic curvature of the deformed generators. In the case of small deformations, $\psi \to w$, the normal displacement (deflection).

(b2)

$$f_{xx} = \frac{1}{a}\left(e_{xx} - \int \gamma_{xs,x}\,\mathrm{d}s\right);\ f_{xs} = 0;$$

$$f_{ss} = \frac{1}{a}e_{ss}. \tag{13b}$$

In the case of small deformations, $\psi_{ss} \to \phi_s$, the circumferential rotation.

(b3)

$$f_{xx} = \frac{1}{a}e_{xx}\,; f_{xs} = \frac{1}{a}\gamma_{xs}\,;\; f_{ss} = \frac{1}{a}e_{ss} + \int \gamma_{xs,s}\,\mathrm{d}x \tag{13c}$$

(b4)

$$f_{xx} = \frac{1}{a}\left(e_{xx} - \int \gamma_{xs,x}\,\mathrm{d}s\right) + \int f_{xs,x}\,\mathrm{d}s\,;$$

$$f_{xs} = -\frac{1}{a}\int e_{ss,x}\,\mathrm{d}s\,;\; f_{ss} = 0. \tag{13d}$$

All of the choices are equally correct and the differences are absorbed in $\psi_{,\alpha\beta}$. Considerations in making the "best" choice for a given class of problems are discussed in Section 2.2.

(c) *Spherical shells*. Here, $B \equiv 0$, so that

$$f_{\alpha\beta} = b^{\mu}_{\mu} e_{\alpha\beta}.$$

(d) *Cylindrical bending of cylindrical surfaces*, including the nonlinear bending of curved beams. Here, $k_{xx} = k_{xs} = e_{xx} = \gamma_{xs} = 0$. There is no dependence on x. In this simple setting, the geometrical interpretation of ψ is apparent. Let $r(s)$ be the circumferential radius of curvature of the shell (or beam). Then, as in case (b), take, for small strains,

$$k_{ss} = \psi_{,ss} + \frac{1}{r}e_{ss}.$$

It is easy to see that, in this case, $\psi_{,s} = \phi$, where ϕ is the *rotation* of cross-sections. Hence,

$$\psi = \int \phi\,\mathrm{d}s = \text{"integral rotation function"}.$$

This simple interpretation emphasizes the difference between ψ and its linearized versions. In fact, for the variation from any *deformed* configuration:

$$\delta\psi_{,s} = \delta\phi = \delta\bar{w}_{,s} + \frac{1}{\bar{r}}\delta\bar{v}_s, \tag{14}$$

where the normal and tangential displacements are measured in the *current normal and tangential directions*. If, for example, $\delta\bar{v}_s = 0$ (normal motion) or is neglected (as in shallow shells), then $\delta\psi = \delta\bar{w}$, and ψ describes *the path of a material point*. It is, in general, unequal to w but approaches it for small displacements. The rigid body rotation of a straight bar is depicted in Fig. 1, with w and ψ shown. The difference is obvious. It is further emphasized in Fig. 1b.

It should be borne in mind that $k_{\alpha\beta}$ are not necessarily equal to the "bending strains" $K_{\alpha\beta}$ which appear in the constitutive relations, nor are they equal to the changes in the physical curvatures $b_{(\alpha\beta)}$. $K_{\alpha\beta}$ are usually related to $k_{\alpha\beta}$ by terms of the $b^{\theta}_{\alpha}e_{\theta\beta}$ type. Of special note is the Sanders–Koiter theory with

$$K_{\alpha\beta} = k_{\alpha\beta} + \tfrac{1}{2}(b^{\theta}_{\alpha}e_{\theta\beta} + b^{\theta}_{\beta}e_{\theta\alpha}). \tag{15}$$

The constitutive relation $n^{\alpha\beta} = \partial U/\partial e_{\alpha\beta}$ needs qualification as to which of the possible bending strain measures are held constant, but all possible $n^{\alpha\beta}$ differ by $b^{\alpha}_{\gamma}m^{\gamma\beta}$ terms (which

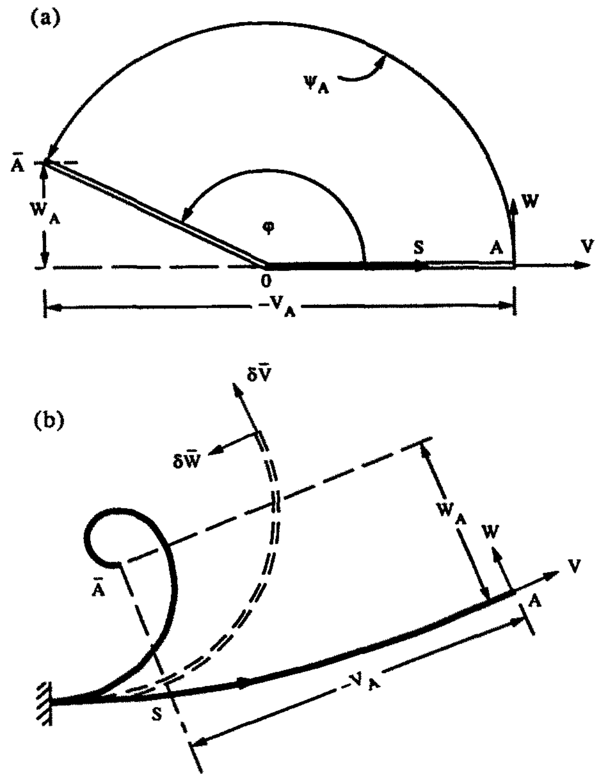

Fig. 1. Cylindrical motion—examples. (a) Rotation of a rod ($\phi = c$, $\psi = cs$). (b) Deformation of a curved beam ($\phi = cs^2$, $\psi = \frac{1}{3}cs^3$).

are of no major significance) and can be converted from one to the other. This study uses the measure

$$n^{\alpha\beta} = \frac{\partial U}{\partial e_{\alpha\beta}}\bigg|_{\psi}, \tag{16}$$

with ψ being held constant.

Using the well known expression for the Gauss compatibility equation in the case of small extensional strains—see, for example, Danielson (1970)—the expression for Π_2 is

$$\Pi_2 = \iint_A \{U - f[\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(e_{\alpha\beta}|_{\gamma\delta} + b_{\alpha\beta}k_{\gamma\delta} + \tfrac{1}{2}k_{\alpha\beta}k_{\gamma\delta}) - Ke^{\alpha}_{\alpha}]\}\,\mathrm{d}A + P, \tag{17}$$

where $k_{\alpha\beta}$ are expressed in terms of ψ and $e_{\alpha\beta}$. Treatment of the external potential P is deferred to the next section.

If $e_{\alpha\beta}$ are set equal to zero in all the terms of eqn (17), then the resulting Π_2 (ψ, f) can be used for the (almost) inextensible analysis of shells. $n^{\alpha\beta}$ become reactive forces, obtainable from separately derived equilibrium equations, e.g. eqns (6.15) and (6.16) of Koiter (1966).

In some problems involving multiple deformation scales, such as small deformation superposed on large, the quadratic strain gradient terms $\beta^{\gamma}_{\alpha\beta}\,\beta^{\lambda}_{\mu\nu}$ in the Gauss equation L_3 [eqn (2.3) of Danielson (1970)] can be retained until after its introduction into Π_2. Only then is Π_2 scaled, retaining quadratic terms in the superposed variables, some of which are multiplied by initial deformation (or stress) parameters. The addition of the above terms is usually of little practical importance, since their effects are small compared with other terms. However, these terms can be of use in *special bifurcation problems* with strong directional effects, such as may arise in long cylindrical shells (see discussion on long tubes in Part II). This topic will not be pursued further in the present paper. For additional discussion, see remark in Section 4(d) of Part II.

2.1. *Shells of weak curvatures*

These are shells having principal radii of curvature R_i, such that $L/R_i \ll 1$, where L is the size of the shell or any deformation length measure on it and R_i is any radius of curvature. Shallow shells are included, as well as all shells having *local* deformation processes. The usual approach to analysis of these shells has been the F–w formulation, where F denotes the stress function and w the normal displacement. This formulation is based on neglecting the tangential displacement terms v_α/R compared with $w_{,\alpha}$ in the expressions for the rotations, thus leading to $k_{\alpha\beta} = w|_{\alpha\beta}$, which is the DMV approximation. Theories and papers based on this formulation are abundant, for example: Alumyae (1949), Libai (1962), Koiter (1966), Pietraszkiewicz (1989), and many more. In fact, Libai (1962) replaced w with a curvature function, thus allowing for stronger nonlinearities. However, the retention of the assumption on the tangential displacements limited the scope of its use. For a more detailed discussion, see Koiter (1966), section 10.

In the present formulation, the geometrical restrictions imposed by $L/R_i \ll 1$ justify the suppression of the extensional strain terms $e_{\alpha\beta}/R$ in the Codazzi equations, leading to $k_{\alpha\beta} = \psi|_{\alpha\beta}$. In the Gauss equation $K = O(R^{-2}) \approx 0$ can be used, but otherwise, all terms involving both $e_{\alpha\beta}$ and ψ are retained. The two approaches lead to equations which are formally similar (with ψ replacing w) but relate to different physical quantities. The fact that approximations are made on the *extensional strains* rather than tangential *displacements* facilitates the application of the present theory to large-displacement, large-rotation analysis (provided that the extensional strains are small). This is in addition to other advantages offered by the mixed variational approach. In the case of small displacements, ψ can be identified with w.

Treatment of these shells in the literature is quite extensive and well developed. They have been used in a wide variety of shell problems, ranging from *local bending* problems to *nonlinear membranes* [the Föppl (1907) problem]. Consequently, their treatment in this paper should be regarded as a *prototype example for the methodology* and not as an end in itself. Using partial integrations, Π_2 reduces in this case to

$$\Pi_2 = \iint_A [U - f\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(e_{\alpha\beta} + b_{\alpha\beta}\psi - \tfrac{1}{2}\psi_{,\alpha}\psi_{,\beta})|_{\gamma\delta}]\,\mathrm{d}A + P_S + P_B. \tag{18}$$

Here, P_S and P_B are surface and boundary potentials, respectively. See below for details.

Consider variations $\delta\bar{v}_\alpha$ and $\delta\bar{w}$ in the normal and tangential displacements from a current (deformed) configuration. Then

$$\delta e_{\alpha\beta} = \tfrac{1}{2}(\delta\bar{v}_\alpha|_\beta + \delta\bar{v}_\beta|_\alpha) - (b_{\alpha\beta} + k_{\alpha\beta})\delta\bar{w}. \tag{19}$$

In the weak curvatures approximation, $\delta\bar{w} = \delta\psi$, $\delta k_{\alpha\beta} = \delta\psi|_{\alpha\beta}$. Thus,

$$\delta e_{\alpha\beta} = \tfrac{1}{2}(\delta\bar{v}_\alpha|_\beta + \delta\bar{v}_\beta|_\alpha) - (b_{\alpha\beta} + \psi|_{\alpha\beta})\delta\psi. \tag{20}$$

Let p^β and p be tangential and normal loadings (in the deformed directions). To incorporate p^β into the functional, $\hat{n}^{\alpha\beta}$ is defined to be any particular symmetric solution of the equations $\hat{n}^{\alpha\beta}|_\alpha + p^\beta = 0$. Then the virtual work of the surface becomes

$$\begin{aligned}\delta P_s &= -\iint_A (p^\beta\delta\bar{v}_\beta + p\delta\bar{w})\,\mathrm{d}A \\ &= -\iint_A \{\hat{n}^{\alpha\beta}\delta e_{\alpha\beta} + [\hat{n}^{\alpha\beta}(b_{\alpha\beta} + \psi|_{\alpha\beta}) + p]\delta\psi\}\,\mathrm{d}A + \int_{\partial L} \hat{n}^{\alpha\beta}\delta\bar{v}_\beta v_\alpha\,\mathrm{d}s,\end{aligned} \tag{21}$$

where $v_\alpha = \varepsilon_{\alpha\beta}\, du^\beta/ds$ are the components of the unit normal to ∂L in the tangent plane.

Since $e_{\alpha\beta}$ are independent fields, the coefficients of $\delta e_{\alpha\beta}$ in $\delta\Pi_2$ must vanish. This yields

$$\left.\frac{\partial U}{\partial e_{\alpha\beta}}\right|_\psi = n^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta} f|_{\gamma\delta} + \hat{n}^{\alpha\beta}. \tag{22}$$

Equation (22) both defines $n^{\alpha\beta}$ and expresses $n^{\alpha\beta}$ and $e_{\alpha\beta}$ in terms of f. It is noted that elimination of f from eqn (22) would yield the tangential equilibrium equations, so that f is a stress function as alleged. The use of eqn (22) for the stress resultants and associated equilibrium equations is *generally accepted* by shell practitioners as an integral part of the weak curvatures approximation of nonlinear shell theory. See, for example, Koiter (1966), sections 10 and 11 and his equations (10.3), (11.48), etc. In the case of (almost) inextensional deformations, eqn (22) is no longer valid since $e_{\alpha\beta} \equiv 0$, but then its deletion from eqn (18) facilitates its use for direct inextensional analysis in terms of ψ and f, as explained before.

Back substitution of $e_{\alpha\beta}(f)$ into eqn (18) yields the modified functional

$$\Pi_3 = \Pi_2'(f, \psi) + P_B + P_S, \tag{23}$$

where Π_2' denotes the surface integral in Π_2. The usefulness of Π_3 is limited by the implicit occurrence of displacements in P_B. However, this difficulty is removed next.

One of the objectives of this paper is to develop a "mixed" principle in which $n^{\alpha\beta}$ and associated quantities are put in a "complementary" form, whereas $m^{\alpha\beta}$ and associated curvatures retain their regular form. As is well known, the admissible stresses in complementary principles must *a priori* satisfy the equilibrium equations and stress boundary conditions. Consequently, any potentials associated with the applied loads are eliminated from the principles.

With this in mind, the following steps are now taken:

(a) Partial integrations are performed on the $-f\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}e_{\gamma\delta}$ terms. The resulting $-n^{\alpha\beta}e_{\alpha\beta}$ terms are joined with U to form a *mixed energy* U_m^* defined by

$$U_m^*(f, \psi) = U - \frac{\partial U}{\partial e_{\alpha\beta}} e_{\alpha\beta} = U_b - U_n^{(c)}, \tag{24}$$

where U_b is the bending energy and $U_n^{(c)}$ is the complementary energy of the force resultants.

(b) The admissible $n^{\alpha\beta}$, defined by eqn (22), are henceforth required to satisfy *a priori* the equations $n^{\alpha\beta}v_\alpha = \bar{T}^\beta$ on ∂L_n, whereas the admissible $k_{\alpha\beta}$ (and associated ψ) are required to satisfy *a priori* the appropriate kinematic conditions on ∂L_v. (The latter are related to changes along the boundary in normal and twisting curvatures.)

(c) The kinematic boundary conditions are assumed to be homogeneous. (Subsequently this condition will be relaxed).

Based on the above considerations, the following mixed functional is defined:

$$\begin{aligned}\Pi^*(f, \psi) &= \iint_A [U_m^* - n^{\alpha\beta}(b_{\alpha\beta}\psi - \tfrac{1}{2}\psi_{,\alpha}\psi_{,\beta})]\, dA + P^* \\ &= \Pi_i^* + P_S^* + P_B^*.\end{aligned} \tag{25}$$

Here, $n^{\alpha\beta}$ is as defined in eqn (22), Π_i^* denotes the internal functional (defined by the surface integral above), and P^* is a mixed external functional consisting of a surface portion P_S^* and a boundary portion P_B^*.

To obtain the Euler–Lagrange equations of the variational equation $\delta\Pi^* = 0$, use is made of the formula

$$\iint_A \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(R_{\alpha\beta}J|_{\gamma\delta}-R_{\beta\alpha}|_{\gamma\delta}J)\,\mathrm{d}A = \oint (R_{\alpha\beta}|_\gamma J - R_{\alpha\beta}J_{,\gamma})\varepsilon^{\beta\gamma}\lambda^\alpha\,\mathrm{d}s$$

for any tensor and scalar functions $R_{\alpha\beta}$ and J, respectively. Proceeding with the calculation of $\delta\Pi_i^*$ the result is:

$$\delta\Pi_i^* = \iint_A \{-[m^{\alpha\beta}|_{\alpha\beta}+n^{\alpha\beta}(b_{\alpha\beta}+\psi|_{\alpha\beta})-p^\beta\psi_{,\beta}]\delta\psi-\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}A_{\alpha\beta}|_{\gamma\delta}\delta f\}\,\mathrm{d}A$$

$$+\oint_{\partial L}[(T^\beta\psi_{,\beta}+Q^\beta v_\beta)\delta\psi-m^{\alpha\beta}\delta\psi_{,\beta}v_\alpha+(A_{\alpha\beta}|_\gamma\delta f-A_{\alpha\beta}\delta f_{,\gamma})\varepsilon^{\beta\gamma}\lambda^\alpha]\,\mathrm{d}s. \quad (26)$$

In the above, $m^{\alpha\beta} = \partial U/\partial k_{\alpha\beta}$, $T^\beta = n^{\alpha\beta}v_\alpha$, $Q^\beta = m^{\alpha\beta}|_\alpha$ and v_α are the components of the unit normal to the boundary line. Also, $A_{\alpha\beta}$ is defined by

$$A_{\alpha\beta} = e_{\alpha\beta}+b_{\alpha\beta}\psi-\tfrac{1}{2}\psi_{,\alpha}\psi_{,\beta}. \quad (26a)$$

For interpretive convenience, consider a local, pseudo-Cartesian coordinate system (n, s) along a boundary, with $u^1 = n$, $u^2 = s$, $\lambda_1 = 0$, $\lambda_2 = 1$, $v_1 = 1$, $v_2 = 0$. In this case, the boundary terms of $\delta\Pi_i^*$ reduce to

$$\oint_{\partial L}[(n^{nn}\psi_{,n}+n^{ns}\psi_{,s}+Q^n+m^{ns}_{,s})\delta\psi-m^{nn}\delta\psi_{,n}$$

$$-(2A_{sn,s}-A_{ss,n})\delta f-A_{ss}\delta f_{,n}]\,\mathrm{d}s+\langle A_{sn}\delta f\rangle-\langle m^{ns}\delta\psi\rangle. \quad (26b)$$

Some notes on the boundary terms are as follows:

(a) The $\langle\cdots\rangle$ terms result from integration by parts along ∂L and denote jumps on the integrands, if any.

(b) The nonlinear terms $T^\beta\psi_{,\beta}\delta\psi$ and $p^\beta\psi_{,\beta}\delta\psi$ reflect the intrinsic formulation, since both T^β and p^β rotate with the deformation.

(c) Intrinsic kinematic quantities associated with ψ and $\psi_{,n}$ are the normal curvature change $\kappa_s = \psi_{,ss}$ and twisting curvature change $\kappa_{ns} = \psi_{,sn}$ along ∂L. If the boundary line is fixed, then $\kappa_s = 0$ (and $\delta\psi = 0$). If it is clamped, then $\kappa_{ns} = 0$ (and $\delta\psi_{,n} = 0$).

(d) The required homogeneous conditions on parts of the boundary ∂L_m, where $\psi_{,n}$ and ψ are not specified, are the vanishing of bending moment m^{nn} (coefficient of $\delta\psi_{,n}$) and effective shear force V^n (coefficient of $\delta\psi$). Corresponding nonhomogeneous conditions are the specification of $\hat{m}_{nn}$ and $\hat{V}^n$.

(e) On parts of the boundary ∂L_n, where forces $\hat{T}^\beta$ are specified, $\delta f = \delta f_{,n} = 0$.

(f) Homogeneous kinematic conditions on parts of the boundary ∂L_v, where forces are not specified, are the vanishing of the coefficient functions:

$$K_\lambda = 2A_{sn,s}-A_{ss,n} = 0 \quad \text{and} \quad A_{ss} = 0.$$

If, in addition, $\psi = 0$ along ∂L_v, then $A_{s\beta} = e_{s\beta}$ and the coefficient functions reduce to the *geodesic curvature change* λ_s and *extensional strain* ε_s of the boundary line. Thus, a fixed boundary line has the conditions $K_s = \lambda_s = \varepsilon_s = 0$. Corresponding nonhomogeneous conditions are the specification of $\hat{K}_\lambda$ and $\hat{A}_{ss}$.

(g) In the case of small finite displacements and moderate rotations at the boundary, $\psi \simeq w$, $\psi_{,\alpha} \simeq w_{,\alpha}$, and

$$A_{\alpha\beta} = e_{\alpha\beta}+b_{\alpha\beta}\psi-\tfrac{1}{2}\psi_{,\alpha}\psi_{,\beta} \simeq \tfrac{1}{2}(v_\alpha|_\beta+v_\beta|_\alpha), \quad (26c)$$

where v_α are the displacements in the tangent plane. The corresponding boundary term reduces (after partial integration) to $\oint v_\beta \delta T^\beta\, ds$. Homogeneous kinematic conditions require the vanishing of v_β, whereas nonhomogeneous conditions involve the specification of $\hat{v}_\beta$.

Consequently, the variation of the external potential for nonhomogeneous boundary data is

$$\delta P^* = -\iint_A (p + p^\beta \psi_{,\beta})\delta\psi\, dA - \oint (\hat{V}_n \delta\psi - \hat{m}^{nn}\delta\psi_{,n} + \hat{K}_\lambda \delta f - \hat{A}_{ss}\delta f_{,n})\, ds - \langle \hat{A}_{sn}\delta f\rangle + \langle \hat{m}^{ns}\delta\psi\rangle. \quad (27)$$

The precise form of P^* in the case of complicated forms of external data is rarely required. In most practical applications, the variation δP^* is used directly. In many problems, the boundary conditions are homogeneous, except for applied forces in the tangent plane at the boundary and normal pressure. In these cases, $\delta P^* = -\iint_A p\delta\psi dA$.

The Euler–Lagrange equations for the variational principle $\delta\Pi^* = 0$ can finally be written as follows:

In A:

$$m^{\alpha\beta}|_{\alpha\beta} + n^{\alpha\beta}(b_{\alpha\beta} + \psi|_{\alpha\beta}) + p = 0 \quad \text{(normal equilibrium)} \quad (28)$$

$$\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(e_{\alpha\beta} + b_{\alpha\beta}\psi - \tfrac{1}{2}\psi_{,\alpha}\psi_{,\beta})|_{\gamma\delta} = 0 \quad \text{(Gauss compatibility)} \quad (29)$$

with $m^{\alpha\beta} = \partial U/\partial k_{\alpha\beta}$, $k_{\alpha\beta} = \psi|_{\alpha\beta}$, and $n^{\alpha\beta}$ and $e_{\alpha\beta}$ expressed in terms of f (and ψ) by eqn (22).

On ∂L, specify

(1) Either ψ (admissibility requirement) or effective shear force $\hat{V}^n = T^\beta\psi_{,\beta} + m^{ns}|_s$
(2) Either $\psi_{,n}$ (admissibility requirement) or bending moment $\hat{m}^{nn} = m^{nn}$
(3) Either T^β (admissibility requirement) or

$$\hat{K}_\lambda = 2A_{ns,s} - A_{ss,n} \text{ and } \hat{A}_{ss} = A_{ss} \quad \text{(in the general case)}$$
$$\hat{\lambda}_s \text{ and } \hat{\varepsilon}_s \text{ (if } \psi \text{ is specified)}$$
$$\hat{v}_n \text{ and } \hat{v}_s \text{ (for small, finite edge deformations).} \quad (30)$$

To these are added jump conditions along the boundary, as described before.

The quantity Π^* is indeed a mixed functional, since it contains a mixed energy with a mixed external functional and also contains mixed "stress–rotation" terms. The latter appear in other functionals, too. See, in particular, those based on the Karman–Marguerre theory of shallow shells: Shih-Ning (1963), Huang (1973) and Gass (1975). The present functional has, however, a much wider scope: it is valid for unrestricted displacements (subject, of course, to the small-strain approximation and the restrictions of weak curvatures) and it contains nonlinear boundary conditions in terms of strains and curvatures.

The development of Π^* for shells of weak curvatures was also intended to serve as an example and prototype for the development of similar principles for other shell systems. For this reason, more details were given in the analysis.

2.2. *Circular cylindrical shells*

The cylindrical shell can serve as a prototype for "quasi-shallow" shells, where $K \cong 0$ but the weak curvatures assumption, $L/R \ll 1$, is not necessarily satisfied, since the shell is not shallow. If "bending theory" dominates throughout the shell, then $f_{\alpha\beta}$ is $O(t/R)$ compared with $k_{\alpha\beta}$ and can be neglected; here t is the shell thickness. In such cases, the results of Section 2.1 can be used. However, in a large number of cases, this assumption is not uniformly valid throughout the shell. In typical cases, bending dominates over a part of the shell, membrane theory, with $k_{\alpha\beta} = O[(1/R)e_{\alpha\beta}]$, over another part, and mixed situations with directional effects are quite common. In fact, some of the difficulties associated with DMV theory can be traced to this. Consequently, a *uniformly valid* representation of $k_{\alpha\beta}$ must incorporate $f_{\alpha\beta}$. For circular cylindrical geometry, eqn (17) becomes

$$\Pi_2 = \iint_A \left[U - f\left(\frac{1}{a}k_{xx} + e_{xx,ss} + e_{ss,xx} - \gamma_{xs,xs} + k_{xx}k_{ss} - k_{xs}^2\right)\right] \mathrm{d}A + P, \tag{31}$$

with $k_{\alpha\beta} = \psi_{,\alpha\beta} + f_{\alpha\beta}$. As explained before, $f_{\alpha\beta}$ are not unique. In making the best choice of $f_{\alpha\beta}$ for a given class of problems, it is important to keep $f_{\alpha\beta}$ to be of the order of the extensional strains $O(\varepsilon)$. A useful guideline is the expected ratio of "differentiation lengths," $\mu = L_x/L_s$, for the given class. In the usual case, $\mu = 0(1)$ and then *any* choice from eqns (13) is acceptable. However, care must be taken in cases with strong directional effects. If $\mu \gg 1$ (as in long tubes and semi-membrane theory), then choices (b) or (d) are appropriate. If $\mu \ll 1$ (edge effect problems), then (a) or (c) must be chosen. In complicated problems, (b) or (c) should be used, since the first derivatives of the strains are still small.

Assuming that $f_{\alpha\beta} = O(\varepsilon)$ and invoking the small extensional strain approximation, one can neglect the quadratic terms in $f_{\alpha\beta}$ in the Gauss equation, compared with the linear extensional strain terms. Then Π_2 reduces to

$$\Pi_2 = \iint_A \left[U - f\left(\frac{1}{a}\psi_{,xx} + \frac{1}{a}f_{xx} + e_{xx,ss} + e_{ss,xx} - \gamma_{xs,xs} + \psi_{,xx}\psi_{,ss} - \psi_{,xs}^2 + \underline{\psi_{,ss}f_{xx}}\right)\right] \mathrm{d}A + P. \tag{32}$$

In the above, the term $\psi_{,xx}f_{ss}$ is $O(\varepsilon)$ compared with $\psi_{,xx}$ and is dropped. The term $\psi_{,xs}f_{xs}$ is $O(\varepsilon)$ compared with either the linear extensional strain terms or $\psi_{,xs}^2$, depending on whether $\psi_{,xs}$ is $O(\varepsilon)$ or $O(1)$. It is $O(\varepsilon^{1/2})$ compared to them if $\psi_{,xs}$ is $O(\varepsilon^{1/2})$. Hence, it may be dropped, too. The underlined term $\psi_{,ss}f_{xx}$ is $O(\varepsilon)$ compared to $\psi_{,xx}$ if $\mu \leqslant O(1)$, but should be retained if *both* $\mu \gg 1$ and $\psi_{,ss} = O(1)$ hold simultaneously (see Part II). Excluding this special case at present, this term is also dropped.

Compared to the weak curvatures case, $f_{\alpha\beta}$ terms are also included in the bending energy U_b. However, only mixed terms $\psi_{\alpha\beta}f_{\gamma\delta}$ need be retained, since quadratic terms in $f_{\alpha\beta}$ are $O[(t/r)^2]$ compared with terms in extensional energy, and can always be neglected. The calculation of δU (using partial integrations on $\delta f_{\alpha\beta}$ as necessary) yields

$$\delta U = n^{\alpha\beta}(e_{\alpha\beta}, \psi)\delta e_{\alpha\beta} - m^{\alpha\beta}(\psi_{,\alpha\beta}, e_{\alpha\beta})\delta\psi_{,\alpha\beta}, \tag{33}$$

with the coefficient functions *defined* to be the force and moment resultants, respectively. The cross-dependence of $n_{\alpha\beta}$ and $m_{\alpha\beta}$ on ψ and $e_{\alpha\beta}$, respectively, comes out of $f_{\alpha\beta}$. Constitutive simplifications may be used according to the problem at hand. As in the general procedure outlined in eqn (3) and as in the case of shells of weak curvature, eqn (22), the coefficients of $\delta e_{\alpha\beta}$ in the variational equation $\delta\Pi_2 = 0$ are to be put equal to zero. This serves both to define $n^{\alpha\beta}$ in terms of f and ψ and to eliminate $e_{\alpha\beta}$ from Π_2. The resulting expressions should depend on the specific choice of f_{xx} in eqns (13). To be more specific, eqn (13a) is chosen for the remainder of this section, with $f_{xx} = 0$ (for another choice of f_{xx} which is useful for long tubes, see Part II, Section 1). With $f_{xx} = 0$ and with the order-of-magnitude analysis as above, the expressions of $n^{\alpha\beta}$ in terms of f are identical to those of eqn (22), but the constitutive form of $n^{\alpha\beta}$ is now

$$n^{\alpha\beta} = n_1^{\alpha\beta}(e_{\gamma\delta}) + n_2^{\alpha\beta}(\psi). \tag{34}$$

Details depend on the form of U for the specific orthotropic material.

Proceeding now as in the case of weak curvatures, a mixed functional Π^* can be constructed. Its form is identical to that of eqn (25). For the specific case of the circular cylindrical shell and the particular choice of $f_{\alpha\beta}$, it becomes

$$\Pi^* = \iint_A \left[U_m^* + n^{ss}\left(\tfrac{1}{2}\psi_{,x}^2 - \frac{1}{a}\psi\right) + \tfrac{1}{2}n^{xx}\psi_{,x}^2 + n^{xs}\psi_{,x}\psi_{,s} \right] \mathrm{d}A + P^*. \tag{35}$$

Consequently, the admissibility requirements, Euler–Lagrange equations, and boundary conditions appear to be formally identical to these in the weak curvatures case. However, the modified constitutive expressions such as eqn (34) are to be used for $m^{\alpha\beta}$ and $n^{\alpha\beta}$, and $k_{\alpha\beta}$ include $f_{\alpha\beta}$ whenever they are used in the boundary conditions. For an example of the boundary conditions in a bent tube, see Part II.

3. CONCLUDING REMARKS

The analysis can be extended to more general quasi-shallow shells without undue difficulties. Spherical and other shells of slowly varying curvatures can use similar procedures. It may be concluded that a "curvature function" embedded in a mixed variational principle shows promise as a tool for large-rotation shell analysis. However, this is limited to cases in which a function ψ which solves the homogeneous Codazzi equations can be found. The classes of shells discussed here are respectable but not exhaustive. Other cases may be added (for example, a trigonometric series form may be used for shells of revolution) but a general solution has yet to be determined. Alternatively, other forms of the compatibility equations may be investigated, but this is beyond the scope of this paper.

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